proposition de *géométrie cinématique*, cherchons d'abord la normale en c' à la surface $[c']$.

Pour un déplacement de l'angle de grandeur constante $a'\,c'\,b'$, le foyer du plan de cet angle est à la rencontre f des normales élevées des points a' et b' à l'ellipsoïde donné. Comme la position de ce foyer est indépendante du déplacement de l'angle mobile, nous en concluons que *la droite* c′ f *est la normale en* c′ *à la surface* [c′].

La bissectrice de l'angle a′ c′ b′ *est toujours la normale en* c′ *à un ellipsoïde* (E) *homofocal à l'ellipsoïde donné et qui contient* c′. Le plan de ces deux normales, c'est-à-dire le plan de l'angle mobile, est alors le plan normal en c' à la ligne d'intersection de (E) et de $[c']$. La tangente à cette courbe d'intersection est alors, comme précédemment, l'un des axes principaux du cône de sommet circonscrit à l'ellipsoïde donné et aussi l'un des axes de l'indicatrice de (E) en c'. Ceci est vrai pour un autre point tel que c' ; nous avons alors ce théorème curieux :

Un angle de grandeur constante, circonscrit à un ellipsoïde donné et dont le plan est normal à cette surface en chacun des points de contact des côtés de cet angle, se déplace de façon que son sommet reste sur l'ellipsoïde(E) *homofocal à l'ellipsoïde donné : ce sommet décrit une ligne de courbure de* (E).*

Ce théorème subsiste si les côtés de l'angle mobile touchent respectivement des ellipsoïdes homofocaux, le plan de cet angle restant toujours normal à ces ellipsoïdes.

On peut encore dire inversement :

Par une tangente à une ligne de courbure d'un ellipsoïde on mène deux plans qui touchent respectivement un ellipsoïde homofocal à celui-ci : l'angle compris entre ces plans est de grandeur constante, quelle que soit cette tangente.

XXIV. "On certain Definite Integrals." No. 9. By W. H. L. Russell, F.R.S. Received June 16, 1881.

Continuing the investigations given in the last paper, we have :—

$$\int_0^{\frac{\pi}{2}} \theta d\theta \{\epsilon^{2i\theta} f(\cos\theta\epsilon^{i\theta}) - \epsilon^{-2i\theta} f(\cos\theta\epsilon^{-i\theta})\} = \pi i \phi \tfrac{1}{2} \quad . \quad . \quad . \quad (176),$$

when $\phi(x) = \int dx\, f(x)$.

$$\int_0^{\pi} \theta d\theta \{\epsilon^{i\theta} f(\epsilon^{i(\theta+\pi)}) - \epsilon^{-i\theta} f(\epsilon^{-i(\theta+\pi)})\} = 2\pi i \phi(1) \quad . \quad . \quad (177),$$

when $\phi(x) = \int dx\, f(x)$ as before.

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* On peut remarquer que la ligne de courbure ainsi décrite rencontre toujours à angle droit le plan de l'angle mobile.

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$$\int_0^\pi d\theta\frac{\sin^2 n\theta}{\sin^2\theta}=n\pi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (178).$$

$$\int_0^\pi d\theta\frac{(2n-1)\sin\theta-\sin(2n-1)\theta}{2\sin^3\theta}=n(n-1)\pi \quad . \; . \; . \; . \; . \; . \; . \quad (179).$$

$$\int_0^\pi d\theta \,.\, \frac{n^2\sin^2\theta-\sin^2 n\theta}{2\sin^4\theta}=\frac{n(n^2-1)\pi}{3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (180).$$

$$\int_0^\infty \frac{\sin^2 n\theta}{\theta\sin\theta}=\frac{n\pi}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (181).$$

$$\int_0^\omega d\theta\frac{(2n-1)\sin\theta-\sin(2n-1)\theta}{\theta\sin^2\theta}=n(n-1)\pi \quad . \; . \; . \; . \; . \; . \; . \; . \quad (182).$$

$$\int_0^\infty d\theta \,.\, \frac{n^2\sin^2\theta-\sin^2 n\theta}{\theta\sin^3\theta}=\frac{n(n^2-1)\pi}{3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (183).$$

$$\int_0^{\frac{\pi}{2}} d\theta \,.\, \frac{1-\cos^n\theta\cos n\theta}{\sin^2\theta}=\frac{n\pi}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (184).$$

From integrals (178), (181), (184), we obtain the three following:—

$$\int_0^\omega d\theta \,.\, \frac{(f(1)-f(\epsilon^{2i\theta}))+(f(1)-f(\epsilon^{-2i\theta}))}{\sin^2\theta}=4\pi f'(1) \quad . \; . \; . \quad (185).$$

$$\int_0^\infty d\theta\frac{(f(1)-f(\epsilon^{2i\theta}))+(f(1)-f(\epsilon^{-2i\theta}))}{\theta\sin\theta}=2\pi f'(1) \quad . \; . \; . \; . \quad (186).$$

$$\int_0^{\frac{\pi}{2}} d\theta \,.\, \frac{(f(1)-f(\cos\theta\,\epsilon^{i\theta})+(f(1)-f(\cos\theta\epsilon^{-i\theta}))}{\sin^2\theta}=\pi f'(1) \quad . \quad (187).$$

$$\int_0^{\frac{\pi}{2}} dx\{^{2ix}f(\cos x\epsilon^{ix})-\epsilon^{-2ix}f(\cos x\epsilon^{-ix})\}=2i\phi(1) \quad . \; . \; . \; . \; . \; . \quad (188).$$

$$\int_0^{\frac{\pi}{2}} d\theta\{\epsilon^{2i\theta}f(\epsilon^{\theta i}\sin\theta)-\epsilon^{-2i\theta}f(\epsilon^{-\theta i}\sin\theta)\}=i\{\epsilon^{\frac{i\pi}{2}}\phi\epsilon^{\frac{i\pi}{2}}+\epsilon^{-\frac{i\pi}{2}}\phi\epsilon^{\frac{i\pi}{2}}\} \quad . \quad (189).$$

$$\int_0^{\frac{\pi}{2}} d\theta\{\epsilon^{2i\theta}f(\sin\theta\epsilon^{i\theta})+\epsilon^{-2i\theta}f(\sin\theta\epsilon^{-i\theta})\}=\frac{\epsilon^{\frac{i\pi}{2}}\phi\epsilon^{\frac{2\pi}{2}}+\epsilon^{-\frac{i\pi}{2}}\phi\epsilon^{-\frac{i\pi}{2}}}{i} \quad . \; . \quad (190).$$

In the three last integrals $\phi(x)$ has the same signification as before.

Let $\rho=a+b\cos\theta+c\cos^2\theta+\;.\;.\;.\;+\cos^n\theta$—

$$\int_0^{\frac{\pi}{2}} d\theta\{f(\rho\epsilon^{ni\theta})+f(\rho\epsilon^{-ni\theta})\}=\pi f\frac{1}{2^n} \quad \ldots \ldots \ldots \quad (191),$$

if $\rho=a+b\sin\theta+\ \ldots\ \sin^n\theta$, where $n=4m+1$.

$$\int_0^{\frac{\pi}{2}} d\theta\{f(\rho\epsilon^{ni\theta})-f(\rho\epsilon^{-ni\theta})\}=i\pi f\frac{1}{2^n} \quad \ldots \ldots \ldots \quad (192).$$

Let α, β, be the roots of $x^2+2x+2=0$, then we have the following integrals:—

$$\int_0^\infty dx\{\epsilon^{\alpha x}f(\epsilon^{\alpha x})+\epsilon^{\beta x}f(\epsilon^{\beta x})\}=\phi(1) \quad \ldots \ldots \ldots \quad (193),$$

$$\int_0^\infty dx\{\epsilon^{\alpha x}f(\epsilon^{\alpha x})-\epsilon^{\beta x}f(\epsilon^{\beta x})\}=i\phi(1) \quad \ldots \ldots \ldots \quad (194).$$

We shall also be able to find:—

$$\int_0^\infty \frac{(a+bx)dx}{g+2hx+x^2}(f\epsilon^{ix})-f(\epsilon^{-ix})) \quad \ldots \ldots \ldots \quad (195).$$

$$\int_0^\infty \frac{(a+bx)dx}{g+2hx+x^2}(f(\epsilon^{ix})+f(\epsilon^{-xi})) \quad \ldots \ldots \ldots \quad (196).$$

Let $\phi_r(x)=\int\int\int\ \ldots\ f(x)dx^r$, then—

$$\int_0^{\frac{\pi}{2}} \theta d\theta\{\epsilon^{4i\theta}f(\cos\theta\epsilon^{i\theta})-\epsilon^{-4i\theta}f(\cos\theta\epsilon^{-i\theta})\}=2i^3\pi\phi_2(\tfrac{1}{2}) \quad \ldots \quad (197).$$

$$\int_0^{\frac{\pi}{2}} \theta d\theta\{\epsilon^{6i\theta}f(\cos\theta\epsilon^{i\theta})-\epsilon^{-6i\theta}f(\cos\theta\epsilon^{-i\theta})\}=8i^5\pi\phi_3(\tfrac{1}{2}) \quad \ldots \quad (198).$$

$$\int_0^\pi \theta d\theta f(\sin\theta)=\frac{\pi}{2}\int_0^\pi d\theta f(\sin\theta) \quad \ldots \ldots \ldots \quad (199).$$

$$\int_0^\infty \frac{d\theta}{\theta}\left\{f\frac{1}{\alpha+\epsilon^{-i\theta}}-f\frac{1}{\alpha+\epsilon^{-i\theta}}\right\}=i\pi\left\{f\frac{1}{1+\alpha}-f0\right\}. \quad \ldots \ldots \quad (200).$$

$$\int_0^\pi d\theta\cos r\theta\left\{\frac{\epsilon^{-\theta i}}{\epsilon^{-\theta i}-\alpha}f\frac{1}{\epsilon^{-\theta i}-\alpha}+\frac{\epsilon^{\theta i}}{\epsilon^{\theta i}-\alpha}f\frac{1}{\epsilon^{\theta i}-\alpha}\right\}$$

$$=\pi\left\{A_0+A_1\frac{d}{d\alpha}+\frac{A_2}{1\,.\,2}\frac{d^2}{d\alpha^2}+\ \ldots\right\}\alpha^r \quad . \quad (201),$$

where $A_0A_1\ \ldots$ are the coefficients of the expansion of $f(x)$.

$$\int_0^{\pi} d\theta \sin r\theta \left\{ \frac{1}{\epsilon^{-\theta i}-\alpha} f \frac{1}{\epsilon^{-\theta i}-\alpha} - \frac{1}{\epsilon^{-\theta i}-\alpha} f \frac{1}{\epsilon^{-\theta i}-\alpha} \right\}$$

$$= 2\pi \left\{ A_0 + A_1 \frac{d}{d\alpha} + \frac{A_2}{1\,.\,2} \cdot \frac{d^2}{d\alpha^2} + \dots \right\} \alpha^{r-1} \quad . \quad (202).$$

$$\int_0^{\infty} \frac{\theta d\theta}{1+\theta^2} \left\{ f \frac{1}{\epsilon^{-\theta i}-\alpha} - f \frac{1}{\epsilon^{-\theta i}-\alpha} \right\} = i\pi \left\{ f \frac{1}{\epsilon-\alpha} - f0 \right\} \quad . \; . \; . \; . \quad (203).$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\tan\theta} \left\{ f \frac{1}{p - i \operatorname{cotan}\theta} - f \frac{1}{p + i \operatorname{cotan}\theta} \right\} = i\pi \left\{ f \frac{1}{p+1} - f0 \right\} \quad . \quad (204).$$

$$\int_0^{\infty} \frac{dx\,.\,x}{q^2+x^2} \left\{ f \frac{1}{p-xi} - f \frac{1}{p+xi} \right\} = i\pi \left\{ f \frac{1}{p+q} - f0 \right\} \quad . \; . \; . \; . \quad (205).$$

$$\int_0^{\infty} \frac{dx}{x} \left\{ f \frac{1}{\rho - ix} - f \frac{1}{\rho + ix} \right\} = i \left\{ f \frac{1}{\rho} - f0 \right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (206).$$

Formula (199) leads to some results, which may well be noticed separately.

$$\int_0^{\pi} \frac{d\theta\,.\,\theta}{a+b\sin\theta} = \frac{\pi}{2\sqrt{a^2-b^2}} \left\{ \pi - \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{b}{\sqrt{a^2-b^2}} \right\} \quad . \; . \quad (207).$$

This may be written (for a particular case) :—

$$\int_0^{\pi} \frac{d\theta\,.\,\theta}{1+2a\sin\theta+a^2} = \frac{\pi}{2(a^2-1)} \left\{ \pi - \frac{2}{a^2-1} \tan^{-1} \frac{2a}{a^2-1} \right\} \quad . \; . \; . \quad (208).$$

$$\int_0^{\pi} \frac{\theta d\theta}{\sqrt{a+b\sin\theta+\dots e\sin^n\theta}} = \frac{\pi}{2} \int_0^{\pi} \frac{d\theta}{\sqrt{a+\dots+e\sin^n\theta}} \quad . \; . \; . \quad (209).$$

Hence
$$\int_0^{\pi} \frac{\theta d\theta}{\sqrt{a+b\sin^2\theta}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (210),$$

can be expressed as an elliptic function.

The integral $\int_0^{\infty} \frac{\tan a\theta}{\theta} = \frac{\pi}{2}$ (stated to be due to Cauchy) which is proved in Gregory's examples, gives us

$$\int_0^{\infty} \frac{d\theta}{\theta} \left\{ \frac{1}{2^n} \operatorname{cotan} \frac{\theta}{2^n} - \operatorname{cotan}\theta \right\} = \frac{\pi}{2}\left(1 - \frac{1}{2^n}\right) \quad . \; . \quad (211).$$

Let (n) increase without limit, then this integral becomes—

$$\int_0^{\infty} \frac{d\theta}{\theta} \left\{ \frac{1}{\theta} - \operatorname{cotan}\theta \right\} = \frac{\pi}{2} \quad . \; . \; . \; . \; . \quad (212).$$

$$\int_0^\pi \frac{d\theta}{\epsilon^{5\alpha\cos\theta}-\epsilon^{3\alpha\cos\theta}+\epsilon^{\alpha\cos\theta}}=\tfrac{1}{2}\int_0^\pi\left(\epsilon^{\alpha\cos\theta}+\epsilon^{-\alpha\cos\theta}\right)d\theta \quad . \quad . \quad (213).$$

Since, in the equation of differences

$$u_{n+2}+pu_{n+1}+qu_n=0,$$

the ratio $\frac{u_{n+1}}{u_n}$ can always be expressed by continued fraction, and since if

$$u_n=\int_\alpha^\beta \frac{x^n}{\sqrt{a+bx+cx^{n2}}},$$

where β and α are the roots $a+bx+cx^2=0$,

$$u_n+\frac{2n-1}{2n}\cdot\frac{b}{c}u_{n-1}+\frac{n-1}{n}\frac{a}{c}u_{n-2}=0,$$

it is evident that the ratio of $\int_\alpha^\beta \frac{x^{r+1}}{\sqrt{a+bx+cx^2}}$ to $\int_\alpha^\beta \frac{x^r}{\sqrt{a+bx+cx^2}}$ can be expressed by a continued fraction. Similar treatment will apply to the integrals—

$$\int\frac{dx\,.\,x^n}{\sqrt[3]{a+bx+cx^2}} \quad . \quad . \quad (214), \qquad \int\frac{x^r dx}{\sqrt{(1-k^2x^2)(1-x^2)}} \quad . \quad . \quad (215),$$

$$\int\frac{x^n}{\sqrt[3]{(1-k^2x^2)(1-x^2)}} \quad (216), \qquad \int\frac{d\theta}{(a+b\cos\theta)^n} \quad . \quad . \quad . \quad . \quad (217),$$

$$\int\frac{d\theta}{(a+b\sin\theta)^n} \quad . \quad . \quad (218), \qquad \int\frac{d\theta}{(1-c^n\sin^2\theta)^{\frac{2n+1}{2}}} \quad . \quad . \quad . \quad (219),$$

and many other integrals.

XXV. "On the Influence of Coal-dust in Colliery Explosions. No. III." By W. GALLOWAY. Communicated by R. H. SCOTT, F.R.S. Received May 30, 1881.

(Abstract.)

The information that can be gleaned from a study of the three great colliery explosions of the past year, namely, Risca, Seaham, and Penygraig, appears to throw more light upon the question than anything that has been elicited experimentally either by myself or others since I last had the honour of addressing the Society on the same subject. These explosions took place on the 15th of July, the 8th of